Published by Institute of Physics Publishing for SISSA/ISAS

Received: *July 8, 2003*
Revised: *August 13, 2003*
Accepted: *August 15, 2003*

On the nature of the finite-temperature transition in QCD

Agostino Butti
Scuola Normale Superiore
Piazza dei Cavalieri 7, I-56126 Pisa, Italy
E-mail: butti@sns.it

Andrea Pelissetto
Dipartimento di Fisica dell'Università di Roma "La Sapienza" and INFN
Piazzale Moro 2, I-00185 Roma, Italy
E-mail: Andrea.Pelissetto@roma1.infn.it

Ettore Vicari
Dipartimento di Fisica dell'Università di Pisa and INFN
Via Buonarroti 2, I-56127 Pisa, Italy
E-mail: vicari@df.unipi.it

Abstract: We investigate the nature of the finite-temperature transition in QCD with N_f massless flavors. Universality arguments show that a continuous phase transition may exist only if there is a stable fixed point in the three-dimensional Φ^4 theory characterized by the symmetry-breaking pattern $[\mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R]/\mathbb{Z}(N_f)_V \to \mathrm{SU}(N_f)_V/\mathbb{Z}(N_f)_V$, or $[\mathrm{U}(N_f)_L \otimes \mathrm{U}(N_f)_R]/\mathrm{U}(1)_V \to \mathrm{U}(N_f)_V/\mathrm{U}(1)_V$ if the $\mathrm{U}(1)_A$ symmetry is effectively restored at T_c. In order to determine the fixed points of these Φ^4 theories, we exploit a three-dimensional perturbative approach in which physical quantities are expanded in powers of renormalized quartic couplings. We compute the perturbative expansion of the β-functions to six loops and determine their large-order behavior. No stable fixed point is found, except for $N_f = 2$ corresponding to the symmetry-breaking pattern $SO(4) \to SO(3)$. Therefore, the finite-temperature phase transition in QCD is of first order for $N_f \geq 3$. A continuous phase transition is allowed only for $N_f = 2$. But, since the theory with symmetry-breaking pattern $[\mathrm{U}(2)_L \otimes \mathrm{U}(2)_R]/\mathrm{U}(1) \to \mathrm{U}(2)_V/\mathrm{U}(1)$ does not have stable fixed points, the transition can be continuous only if the effective breaking of the $\mathrm{U}(1)_A$ symmetry is sufficiently large.

Keywords: Renormalization Group, Field Theories in Lower Dimensions, QCD, Lattice QCD.

http://jhep.sissa.it/archive/papers/jhep082003029/jhep082003029.pdf

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1. Introduction

The thermodynamics of quarks and gluons described by QCD is characterized by a transition from a low-temperature hadronic phase, in which chiral symmetry is broken, to a high-temperature phase with deconfined quarks and gluons (quark-gluon plasma), in which chiral symmetry is restored. The main features of this transition depend crucially on the QCD parameters, such as the number N_f of flavors and the quark masses. In the light-quark regime the nature of the finite-temperature transition is essentially related to the restoring of the chiral symmetry [1, 2, 3, 4]. For recent reviews see, e.g., refs. [5, 6, 7, 8].

In the vanishing quark-mass limit, the QCD lagrangian is invariant under $\mathrm{U}(N_f)_L$ and $\mathrm{U}(N_f)_R$ transformations involving the left- and right-handed quark flavors. Since $\mathrm{U}(N)_{L,R} \cong \mathrm{U}(1)_{L,R} \otimes [\mathrm{SU}(N)/\mathbb{Z}(N)]_{L,R}$ and the group $\mathrm{U}(1)_L \otimes \mathrm{U}(1)_R$ is isomorphic to the group $\mathrm{U}(1)_V \otimes \mathrm{U}(1)_A$ of vector and axial $\mathrm{U}(1)$ transformations, the classical symmetry group of the theory can be written as

$$\mathrm{U}(1)_V \otimes \mathrm{U}(1)_A \otimes [\mathrm{SU}(N_f)/\mathbb{Z}(N_f)]_L \otimes [\mathrm{SU}(N_f)/\mathbb{Z}(N_f)]_R \tag{1.1}$$

The vector subgroup $\mathrm{U}(1)_V$ corresponds to the quark-number conservation, and it is not expected to play any role in the transition. The axial subgroup $\mathrm{U}(1)_A$ is broken by the anomaly at the quantum level, reducing the relevant symmetry to [1] $[\mathrm{SU}(N_f)_L \otimes$

$\mathrm{SU}(N_f)_R]/\mathbb{Z}(N_f)_V$. At $T = 0$ the symmetry is spontaneously broken to $\mathrm{SU}(N_f)_V$ with $N_f^2 - 1$ Goldstone particles (pions and kaons) with a nonzero quark condensate $\langle\bar{\psi}\psi\rangle$. With increasing T, a phase transition characterized by the restoring of the chiral symmetry is expected at T_c; above T_c the quark condensate vanishes. The symmetry-breaking pattern at the transition is expected to be

$$[\mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R]/\mathbb{Z}(N_f)_V \to \mathrm{SU}(N_f)_V/\mathbb{Z}(N_f)_V \,, \tag{1.2}$$

with order parameter given by the expectation value of the quark bilinear $\Psi_{ij} \equiv \bar{\psi}_{L,i}\psi_{R,j}$. In this picture the quark masses act as an external field H_{ij} coupled to the order parameter. Note that, if the $\mathrm{U}(1)_A$ symmetry is effectively restored at T_c, the relevant symmetry-breaking pattern would be

$$[\mathrm{U}(N_f)_L \otimes \mathrm{U}(N_f)_R]/\,\mathrm{U}(1)_V \to \mathrm{U}(N_f)_V/\,\mathrm{U}(1)_V \,. \tag{1.3}$$

The effects of the anomaly tend to be suppressed in the large-N_c limit, where N_c is the number of colors. The symmetry-breaking pattern (1.3) should be recovered in the limit of an infinite number of colors.

In the case of a continuous transition, thermodynamic quantities are analytic functions of T as long as the external field coupled to the order parameter is nonvanishing. Therefore, if the chiral-symmetry restoring transition is continuous in the vanishing quark-mass limit, an analytic crossover is expected for nonvanishing quark masses (actually for not too large values because in the heavy-quark limit the transition is first order [9]). On the other hand, if the phase transition is of first order in the chiral limit, it persists also for nonvanishing masses, up to a critical surface in the quark-mass phase diagram where the transition becomes continuous and is expected to be Ising-like [4]. Outside this surface, i.e. for larger quark masses, the phase transition disappears and we have an analytic crossover.

The nature of the finite-temperature transition in QCD with N_f light flavors has been much investigated, exploiting various approaches, including numerical simulations of lattice QCD. Our understanding is still essentially based on the universality arguments reported by Pisarski and Wilczek in ref. [1]. Their main points can be summarized as follows:

(i) Let us first assume that the phase transition at T_c is continuous (second order) for vanishing quark masses. In this case the critical behavior should be described by an effective three-dimensional (3-d) theory. Indeed, the length scale of the critical modes diverges approaching T_c, becoming eventually much larger than $1/T_c$ that is the size of the euclidean "temporal" dimension at T_c. Therefore, the asymptotic critical behavior must be associated with a 3-d universality class characterized by a complex $N_f \times N_f$ matrix order parameter, corresponding to the bilinear $\Psi_{ij} = \bar{\psi}_{L,i}\psi_{R,j}$, and by the symmetry-breaking pattern (1.2), or (1.3) if the $\mathrm{U}(1)_A$ symmetry is effectively restored at T_c.

(ii) According to renormalization-group (RG) theory, the existence of such universality classes can be investigated by considering the most general Landau-Ginzburg-Wilson (LGW) Φ^4 theory for a complex $N_f \times N_f$ matrix field Φ_{ij} with the desired symmetry

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and symmetry-breaking pattern. The LGW lagrangian for the symmetry-breaking pattern (1.3) is given by

$$\mathcal{L}_{\mathrm{U}(N_f)} = \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r\,\mathrm{Tr}\,\Phi^\dagger\Phi + \frac{u_0}{4}\left(\mathrm{Tr}\,\Phi^\dagger\Phi\right)^2 + \frac{v_0}{4}\,\mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2 . \tag{1.4}$$

The chiral anomaly reduces the symmetry to (1.2) and thus new terms must be added. The most relevant one is proportional to $\left(\det\Phi^\dagger + \det\Phi\right)$, which is a polynomial of order N_f in the field Φ. Thus, for $N_f \geq 5$ such a term, and therefore the $\mathrm{U}(1)_A$-symmetry breaking, is irrelevant at the transition. Instead, for $N_f = 3$ and $N_f = 4$ the new term must be added to the lagrangian (1.4), obtaining

$$\mathcal{L}_{\mathrm{SU}(N_f)} = \mathcal{L}_{\mathrm{U}(N_f)} + w_0\left(\det\Phi^\dagger + \det\Phi\right). \tag{1.5}$$

For $N_f \leq 2$ additional terms depending on the determinant of Φ should be added, in order to include all terms compatible with the symmetry and with at most four fields. Nonvanishing quark masses can be accounted for by adding an external-field term H_{ij}, such as

$$\mathrm{Tr}\left(H\Phi + \mathrm{h.c.}\right). \tag{1.6}$$

(iii) The critical behavior at a continuous transition is described by the stable fixed point (FP) of the theory, which determines the universality class. The absence of a stable FP indicates that the phase transition is not continuous. Therefore, a necessary condition of consistency with the initial hypothesis that the transition is continuous, cf. (i), is the existence of stable FP's in the theories described by the lagrangians (1.4) and (1.5).

(iv) In the one-flavor case, $N_f = 1$, the lagrangian $\mathcal{L}_{\mathrm{U}(N_f)}$ is equivalent to the O(2)-symmetric real Φ^4 lagrangian, which has a stable FP describing the XY universality class. For $N_f = 1$ the determinant term in the lagrangian $\mathcal{L}_{\mathrm{SU}(N_f)}$ is equivalent to an external field coupled to the order parameter, which smooths out the singularity of the continuous transition.

In the two-flavor case, $N_f = 2$, the symmetry-breaking pattern (1.2) is equivalent to $\mathrm{SO}(4) \to \mathrm{SO}(3)$, which is the symmetry-breaking pattern of the standard SO(4)-symmetric theory for a four-dimensional vector field. As is well known, such a theory has a stable FP, corresponding to the SO(4) universality class. Thus, for $N_f = 2$, there exists a three-dimensional universality class with the desired symmetry-breaking pattern. This fact tells us that, if the phase transition is continuous, it belongs to the SO(4) universality class, i.e. it has the same critical exponents, scaling functions, critical equation of state, etc.... However, these arguments do not exclude that the transition is a first-order one.

(v) The problem is more complex in the other cases, i.e. $N_f \geq 3$ for the symmetry breaking (1.2) and $N_f \geq 2$ for the symmetry breaking (1.3). In order to investigate the RG flow of the lagrangians (1.4) and (1.5), Pisarski and Wilczek [1] performed a first-order perturbative calculation within the ϵ-expansion framework, i.e. an expansion in

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powers of $\epsilon \equiv 4 - d$. Within this approximation, they did not find stable FP's. Then, extending this result to three dimensions, i.e. to $\epsilon = 1$, they argued that the finite-temperature transition in QCD is always a first-order one, with the only possible exception of $N_f = 2$.

The ϵ expansion provides useful indications for the behavior of the RG flow in three dimensions. But the validity of the extrapolation to $\epsilon = 1$ of the results obtained near four dimensions is not guaranteed, not even at a qualitative level, especially if they are obtained from low-order computations. The location and the stability of the FP's may drastically change approaching $d = 3$, and new FP's, not present for $d \approx 4$, may appear in three dimensions. In several physically interesting cases low-order (and in some cases also high-order) ϵ-expansion calculations fail to provide the correct physical picture. We mention the 3-d O(N)$\oplus$O(M) invariant Φ^4 theory describing the multicritical behavior arising from the competition of distinct order parameters [10, 11],[1] the 3-d O(N)$\otimes$O(M) Φ^4 theory describing the critical behavior of frustrated spin models with noncollinear order [12, 13],[2] and the Ginzburg-Landau model of superconductors, in which a complex scalar field couples to a gauge field [14, 15]. For example, in the latter case one-loop ϵ-expansion calculations [14] indicate that no stable FP exists unless the number N of real components of the scalar field is larger than $N_c = 365$. This number is much larger than the physical value $N = 2$. Consequently, a first-order transition was always expected [14]. Later, exploiting three-dimensional theoretical approaches (see, e.g., ref. [15]) and Monte Carlo simulations (see, e.g., ref. [16]), it was realized that three-dimensional systems described by the Ginzburg-Landau model can also undergo a continuous transition — this implies the presence of a stable FP in the 3-d Ginzburg-Landau theory — in agreement with experiments [17]. Therefore, a three-dimensional analysis, not relying on extrapolations from $\epsilon \ll 1$ to $\epsilon = 1$, is called for in order to check the picture reported at point (v) above.

In this paper we return to point (v), i.e. to the issue concerning the existence of three-dimensional universality classes with the symmetry-breaking pattern of QCD, and therefore the existence of stable FP's in the theories defined by the lagrangians (1.4) and (1.5). In order to obtain a reliable picture of the RG flow, we perform a calculation in a perturbative framework in which the dimension d is fixed to the physical value, i.e. $d = 3$, and the expansion is performed in powers of appropriate renormalized quartic couplings. We compute the corresponding Callan-Symanzik β-functions to six loops. Moreover, we study the general properties of the expansions: we determine the region in which they are Borel summable and compute their large-order behavior to further constrain the analysis. This allows us to reconstruct the 3-d RG flow, and in particular to check the existence of FP's, with a quite high confidence. We show that no stable FP exists in all cases considered at point (v) above, i.e. $N_f \geq 3$ for the symmetry breaking (1.2) and $N_f \geq 2$ for the symmetry breaking (1.3). Therefore, we fully confirm the conclusions of ref. [1], putting them on a firmer ground.

[1]Particularly important cases are $N = 1$, $M = 2$, relevant for anisotropic antiferromagnets, and $M = 2$, $N = 3$ relevant for high-T_c superconductors.

[2]These models are physically realized by stacked triangular antiferromagnets, in the case $N = 2$, $M = 2, 3$.

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Concerning QCD, these results indicate that the finite-temperature phase transition is of first order for $N_f \geq 3$. A continuous transition is allowed only for $N_f = 2$. But, since no stable FP is associated with the symmetry breaking $[\mathrm{U}(2)_L \otimes \mathrm{U}(2)_R]/\,\mathrm{U}(1)_V \to \mathrm{U}(2)_V/\,\mathrm{U}(1)_V$, the transition can be continuous only if the effective breaking of the $\mathrm{U}(1)_A$ symmetry is sufficiently large. Otherwise, it is first order. For a specific effective breaking of the $\mathrm{U}(1)_A$ symmetry, we may also have a mean-field critical behavior with logarithmic corrections, which is associated with the tricritical point that separates the first-order and the second-order transition lines in the $T - g$ plane, where g parametrizes the effective breaking of the $\mathrm{U}(1)_A$ symmetry.

Direct information on the QCD phase transitions has been provided by numerical Monte Carlo studies within the lattice formulation of QCD, see, e.g., the recent review [8]. The Monte Carlo results obtained so far are consistent with and substantially support the above-reported three-dimensional RG results. In the case of three-flavor QCD the evidence that the transition is first order is rather clear, see, e.g., refs. [18]–[22]. Moreover, it has been verified that the first-order transition persists for nonvanishing quark masses up to Ising-like endpoints [18]. In the two-flavor case recent lattice simulations have provided a rather convincing evidence of a continuous transition at $T_c \approx 172\,\mathrm{MeV}$, see, e.g., refs. [23]–[27], whose scaling properties are substantially consistent with those predicted by the 3-d SO(4) universality. However, a conclusive evidence in favor of an SO(4) scaling behavior in the continuum limit has not been achieved yet. Apparently, a mean-field critical behavior has not been ruled out yet. We also mention that the effective $\mathrm{U}(1)_A$-symmetry breaking at finite temperature has been investigated on the lattice. The $\mathrm{U}(1)_A$ symmetry appears not to be restored at T_c, as also suggested by instanton calculations [28], although a significant suppression of the effective breaking is observed above T_c, see, e.g., refs. [29, 30, 31].

The paper is organized as follows. In section 2 we present the general three-dimensional perturbative framework. In section 3 we report our calculations for a theory with symmetry $[\mathrm{U}(N)_L \otimes \mathrm{U}(N)_R]/\,\mathrm{U}(1)_V$, i.e. for the lagrangian (1.4).[3] The analysis of the perturbative expansions, using also information on their large-order behavior, shows no evidence of stable FP's, for any N. In section 4 we consider the more general lagrangian (1.5), which is symmetric under the smaller group $[\mathrm{SU}(N)_L \otimes \mathrm{SU}(N)_R]/\mathbb{Z}(N)_V$. We present a six-loop analysis for $N = 4$, which is the only case for which no convincing arguments exist in favor or against the existence of a stable FP. Again, our analysis does not find evidence of a stable FP. Finally, in the appendix we discuss in detail the phase diagram of the general Φ^4 model realizing the symmetry-breaking pattern (1.2) for $N = 2$ with a complex 2×2 matrix order parameter, which is relevant for the finite-temperature transition in two-flavor QCD.

2. Perturbative expansion in Landau-Ginzburg-Wilson Φ^4 theories

2.1 General Φ^4 theories

In the RG approach to critical phenomena, many phase transitions can be investigated by considering effective LGW theories, containing up to fourth-order powers of the field

[3] For simplicity, from now on we simply write N instead of N_f.

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components. The general LGW lagrangian for an N-component field ϕ_i can be written as

$$\mathcal{L}_{\rm LGW} = \int d^d x \left[\frac{1}{2} \sum_i (\partial_\mu \phi_i)^2 + \frac{1}{2} \sum_i r_i \phi_i^2 + \frac{1}{4!} \sum_{ijkl} u_{ijkl} \phi_i \phi_j \phi_k \phi_l \right] , \tag{2.1}$$

where the number of independent parameters r_i and u_{ijkl} depends on the symmetry group of the theory. An interesting class of models are those in which $\sum_i \phi_i^2$ is the only quadratic invariant polynomial. In this case, all r_i are equal, $r_i = r$, and u_{ijkl} satisfies the trace condition [32]

$$\sum_i u_{iikl} \propto \delta_{kl} \, . \tag{2.2}$$

In these models, criticality is driven by tuning the single parameter r. Therefore, they describe critical phenomena characterized by one (parity-symmetric) relevant parameter, which often corresponds to the temperature. Of course, there is also (at least one) parity-odd relevant parameter that corresponds to a term $\sum_i h_i \phi_i$ that can be added to the hamiltonian (2.1). For symmetry reasons, criticality occurs for $h_i \to 0$. The simplest example of this class of models is the O(N)-symmetric Φ^4 theory, which has only one quartic parameter u:

$$u_{ijkl} = \frac{u}{3} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) . \tag{2.3}$$

This model describes several interesting universality classes: the Ising one for $N = 1$ (e.g., liquid-vapor transition, transitions in uniaxial magnetic systems, etc...), the XY one for $N = 2$ (e.g., superfluid transition in ^{4}He, transitions in magnets with easy-plane anisotropy, etc...), the Heisenberg one for $N = 3$ (e.g., Curie transition in isotropic magnets); for $N \to 0$ it describes the behavior of dilute homopolymers for large degree of polymerization. See, e.g., refs. [33, 34] for recent reviews. But there are also other physically interesting transitions that are characterized by more complex symmetries: for instance, the magnetic transitions in randomly dilute magnetic systems, in stacked triangular antiferromagnets, in Mott insulators, and several structural phase transitions. See, e.g., refs. [35, 34, 51, 52] for reviews. In this case, the corresponding LGW lagrangian contains two or more quartic couplings, but still only one quadratic invariant. More general LGW lagrangians, that allow for the presence of independent quadratic parameters r_i, have been considered to study the multicritical behavior that arises from the competition of distinct types of ordering, see, e.g., refs. [10, 11].

In the field-theory (FT) approach the RG flow is determined by a set of RG equations for the correlation functions of the order parameter. In the case of a continuous transition, the critical behavior is determined by the stable FP of the theory, which characterizes a universality class. The absence of a stable FP is usually considered as an indication that the phase transition is of first order, even in those cases in which the mean-field approximation predicts a continuous transition. But, even in the presence of a stable FP, a first-order transition may still occur for systems that are outside its attraction domain.

The RG flow can be studied by perturbative methods, for example by performing an expansion in powers of $\epsilon \equiv 4 - d$ or a fixed-dimension (FD) expansion in powers of appropriate zero-momentum renormalized quartic couplings. The ϵ expansion, which was introduced by Wilson and Fisher [36], is based on the observation that, for $d = 4$, the theory is essentially gaussian. One considers the standard perturbative expansion, and then transforms it into an expansion in powers of $\epsilon \equiv 4 - d$. Subsequently, Parisi [37] pointed out the possibility of using perturbation theory directly at the physical dimensions $d = 3$ and $d = 2$ in the massive (high-temperature) phase. In the following we will follow the latter approach.

2.2 The fixed-dimension expansion

Let us consider the generic Φ^4 lagrangian (2.1) for an N-component real field ϕ_i, with a single parameter r, i.e. $r_i = r$ for all r, so that the quartic terms satisfy the trace condition (2.2). In the framework of the FD expansion one works directly in the dimension of interest, $d = 3$ or $d = 2$. In this case the theory is super-renormalizable, since the number of primitively divergent diagrams is finite. One may regularize the corresponding integrals by keeping d arbitrary and performing an expansion in $\varepsilon = 3 - d$ or $\varepsilon = 2 - d$. Poles in ε appear in divergent diagrams. Such divergences are related to the necessity of performing an infinite renormalization of the parameter r appearing in the bare lagrangian, see, e.g., the discussion in ref. [38]. This problem can be avoided by replacing r with the mass m defined by

$$m^{-2} = \frac{1}{\Gamma^{(2)}(0)} \left. \frac{\partial \Gamma^{(2)}(p^2)}{\partial p^2} \right|_{p^2=0}, \tag{2.4}$$

where the function $\Gamma^{(2)}(p^2)$ is related to the one-particle irreducible two-point function by

$$\Gamma^{(2)}_{ij}(p) = \delta_{ij} \Gamma^{(2)}(p^2) \,. \tag{2.5}$$

Perturbation theory in terms of m and u_{ijkl} is finite. The critical limit is obtained for $m \to 0$. To handle it, one considers appropriate RG functions. Specifically, one defines the renormalized four-point couplings g_{ijkl} and the field-renormalization constant Z_ϕ by

$$\Gamma^{(2)}_{ij}(p) = \delta_{ij} Z_\phi^{-1} \left[m^2 + p^2 + O(p^4) \right] , \tag{2.6}$$

$$\Gamma^{(4)}_{ijkl}(0) = m^{4-d} \, Z_\phi^{-2} \, g_{ijkl} \,, \tag{2.7}$$

where $\Gamma^{(n)}_{a_1,\ldots,a_n}$ are n-point one-particle irreducible correlation functions. Eqs. (2.6) and (2.7) relate the mass m and the renormalized quartic couplings g_{ijkl} to the corresponding bare lagrangian parameters r and u_{ijkl}. In addition, one introduces the function Z_t that is defined by the relation $\Gamma^{(1,2)}_{ij}(0) = \delta_{ij} Z_t^{-1}$, where $\Gamma^{(1,2)}$ is the one-particle irreducible two-point function with an insertion of $\frac{1}{2} \sum_i \phi_i^2$.

The common zeroes of the Callan-Symanzik β-functions

$$\beta_{ijkl}(g_{abcd}) = m \left. \frac{\partial g_{ijkl}}{\partial m} \right|_{u_{abcd}} \tag{2.8}$$

provide the FP's of the theory. In the case of a continuous transition, when $m \to 0$, the couplings g_{ijkl} are driven towards an infrared-stable FP. The stability properties of the

FP's are controlled by the eigenvalues ω_i of the matrix

$$\Omega_{ijkl,abcd} = \frac{\partial \beta_{ijkl}}{\partial g_{abcd}} \tag{2.9}$$

computed at the given FP: a FP is stable if all eigenvalues ω_i have positive real part. The critical exponents are then obtained by evaluating the RG functions

$$\eta_\phi(g_{ijkl}) = \left.\frac{\partial \ln Z_\phi}{\partial \ln m}\right|_u, \qquad \eta_t(g_{ijkl}) = \left.\frac{\partial \ln Z_t}{\partial \ln m}\right|_u \tag{2.10}$$

at the stable FP g^*_{ijkl}: $\eta = \eta_\phi(g^*_{ijkl})$, and $\nu = [2 - \eta_\phi(g^*_{ijkl}) + \eta_t(g^*_{ijkl})]^{-1}$. From the perturbative expansion of the correlation functions $\Gamma^{(2)}$, $\Gamma^{(4)}$, and $\Gamma^{(1,2)}$ and using the above-reported relations, one can derive the expansion of the RG functions β_{ijkl}, η_ϕ, and η_t in powers of g_{ijkl}.

2.3 Resummation of the series

FT perturbative expansions are divergent. Thus, in order to obtain accurate results, an appropriate resummation is required. In the case of the O(N)-symmetric models, accurate results have been obtained from the analysis of the available FD six-loop series [39] by exploiting Borel summability (proved in dimension $d < 4$ [40]) and the knowledge of the large-order behavior of the expansion, see, e.g., refs. [41, 42]. If we consider a quantity $S(g)$ that has a perturbative expansion[4]

$$S(g) \approx \sum s_k g^k\,, \tag{2.11}$$

the large-order behavior of the coefficients is generally given by

$$s_k \sim k!\,(-a)^k\,k^b\left[1 + O(k^{-1})\right]. \tag{2.12}$$

Borel summability of the series for $g > 0$ requires $a > 0$. The value of the constant a is independent of the particular quantity considered, unlike the constant b. They can be determined by means of a steepest-descent calculation in which the saddle point is a finite-energy solution (instanton) of the classical field equations with negative coupling [43, 44], see also refs. [33, 45].[5] In three dimensions and for any N, $a = 0.00881962\ldots$, see, e.g., ref. [33]. In order to resum the perturbative series, one may introduce the Borel-Leroy transform $B(t)$ of $S(g)$,

$$S(g) = \int_0^\infty dt\, t^c e^{-t} B(gt)\,, \tag{2.13}$$

where c is an arbitrary number. Its expansion is given by

$$B_{\rm exp}(t) = \sum_k \frac{s_k}{\Gamma(k+c+1)} t^k\,. \tag{2.14}$$

[4]Here the coupling g is normalized according to eq. (2.7) with $g_{ijkl} = \frac{1}{3}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) g$.

[5]More precisely, the saddle-point $f_s(x)$ is a solution of the equation $\Delta f_s = f_s - f_s^3$. One can show that a is related to its classical action, indeed $a = -1/S_f$ where $S_f = -\frac{3}{2}\int d^dx\, f_s^4$ [33].

The constant a that characterizes the large-order behavior of the original series is related to the singularity t_s of the Borel transform $B(t)$ that is nearest to the origin: $t_s = -1/a$. The series $B_{\rm exp}(t)$ is convergent in the disk $|t| < |t_s| = 1/a$ of the complex plane, and also on the boundary if $c > b$. In this domain, one can compute $B(t)$ using $B_{\rm exp}(t)$. However, in order to compute the integral (2.13), one needs $B(t)$ for all positive values of t. It is thus necessary to perform an analytic continuation of $B_{\rm exp}(t)$. For this purpose one may use Padé approximants to the series (2.14) [39]: This is the so-called Padé-Borel resummation method. A more refined procedure exploits the knowledge of the large-order behavior of the expansion, and in particular of the constant a in eq. (2.12). One performs an Euler transformation [41]

$$y(t) = \frac{\sqrt{1+at}-1}{\sqrt{1+at}+1}\,, \tag{2.15}$$

that allows to rewrite $B(t)$ in the form

$$B(t) = \sum_k f_k \, [y(t)]^k \,. \tag{2.16}$$

If all the singularities of $B(t)$ belong to the real interval $[-\infty, -t_c]$, the expansion (2.16) converges everywhere in the complex t-plane except on the negative axis for $t < -t_c$. After these transformations, one obtains a new expansion for the function $S(g)$:

$$S(g) \approx \int_0^\infty dt \, e^{-t} t^c \sum_k f_k \, [y(tg)]^k \,. \tag{2.17}$$

This sequence of operations has transformed the original divergent series into an integral of a convergent one, which can then be studied numerically. However, the convergence of the integral (2.17), that is controlled by the analytic properties of $S(g)$, is not guaranteed. Indeed, since generic RG quantities have a cut for $g \geq g^*$ [37, 46, 47], where g^* is the FP, the integral does not converge for $g > g^*$.

The method can be extended to the case of more general LGW theories, with several quartic couplings, see, e.g., refs. [48, 34]. In the case of k independent quartic lagrangian terms, a generic quantity S depends on k renormalized couplings. Its expansion can be written as

$$S(g_1, \ldots, g_k) = \sum_{j_1,\ldots,j_k} c_{j_1,\ldots,j_k} g_1^{j_1} \ldots g_k^{j_k} = \sum_n s_n g^n \,, \tag{2.18}$$

where

$$s_n = \sum_{j_1+\cdots+j_k=n} c_{j_1,\ldots,j_k} \hat{g}_1^{j_1} \ldots \hat{g}_k^{j_k} \,, \qquad \hat{g}_j = \frac{g_j}{g} \,. \tag{2.19}$$

The large-order behavior of the coefficients s_n, and therefore the singularities of the Borel transform, depends on the ratios $\hat{g}_i$, but it can be still determined by steepest-descent calculations. Borel summability requires that the Borel trasform does not have singularities on the positive real axis. Under the assumption that the group and the space structure of the relevant saddle points are decoupled,[6] it is possible to show that the region where Borel

[6]This fact has been proved to hold in O(N)-symmetric models [44].

summability holds is directly related to the region of stability of the lagrangian potential with respect to the bare quartic couplings: with the normalizations implicit in eq. (2.7), it is enough to replace the bare quartic couplings with their renormalized counterparts. For instance, in $O(N)$-symmetric models the potential is stable only for $u > 0$ and therefore the expansion is Borel summable only for $g > 0$. Once the large-order behavior of the coefficients s_n has been determined, i.e.

$$s_n \sim n!\,[-A(\hat{g}_1, \dots, \hat{g}_k)]^n\, n^b\,, \tag{2.20}$$

one may use a conformal-mapping method based on eq. (2.15) to resum the series in g, as in the $O(N)$-symmetric case. Again, $A(\hat{g}_1, \dots, \hat{g}_k)$ does not depend on the quantity at hand. Alternatively, one may use the Padé-Borel method, employing Padé approximants to analytically extend the Borel transform.

3. The $U(N)_L \otimes U(N)_R$ effective theory

3.1 The effective LGW Φ^4 theory

The most general Φ^4 lagrangian that is symmetric under $U(N)_L \otimes U(N)_R$ transformations is given by

$$\mathcal{L}_{U(N)} = \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r\, \mathrm{Tr}\, \Phi^\dagger \Phi + \frac{u_0}{4}\left(\mathrm{Tr}\, \Phi^\dagger \Phi\right)^2 + \frac{v_0}{4}\, \mathrm{Tr}\left(\Phi^\dagger \Phi\right)^2, \tag{3.1}$$

where the fundamental field Φ_{ij} is a complex $N \times N$ matrix. There is only one quadratic invariant and therefore the trace condition (2.2) is satisfied by the quartic terms. The lagrangian (3.1) is invariant under the transformations $\Phi \to A\Phi B$, A, $B \in U(N)$, and the corresponding symmetry group is $[U(N)_L \otimes U(N)_R]/\,U(1)_V$. The condition $v_0 > 0$ is necessary to realize the symmetry breaking

$$[U(N)_L \otimes U(N)_R]/\,U(1)_V \to U(N)_V/\,U(1)_V\,, \tag{3.2}$$

where $U(N)_V$ is the group of transformations $\Phi \to A\Phi A^\dagger$, $A \in U(N)$. Indeed, for $r < 0$ and $v_0 > 0$ the minimum of the quartic potential, which has the form $\Phi_{ij} = \phi_0 \delta_{ij}$, is symmetric under $U(N)$ transformations, thus leading to the desired symmetry-breaking pattern. For $v_0 < 0$ the minimum of the potential is obtained for $\Phi_{ij} = \phi_0 \delta_{ia} \delta_{ja}$, where a is an arbitrary integer, $1 \leq a \leq N$. As one can easily check, this leads to a different symmetry-breaking pattern. Note that for $N = 1$ the two quartic terms are equal, and the lagrangian $\mathcal{L}_{U(N)}$ becomes equivalent to the standard $U(1)$-symmetric Φ^4 lagrangian for a complex field with coupling $u_0 + v_0$, which in turn is directly related to the $O(2)$-symmetric real Φ^4 lagrangian.

As explained in section 2.2, the theory is renormalized by introducing a set of zero-momentum conditions for the one-particle irreducible two-point and four-point correlation functions:

$$\Gamma^{(2)}_{a_1 a_2, b_1 b_2}(p) = \frac{\delta^2 \Gamma}{\delta \Phi^\dagger_{a_1 a_2} \delta \Phi_{b_1 b_2}} = \delta_{a_1 b_1} \delta_{a_2 b_2} Z_\phi^{-1} \left[m^2 + p^2 + O(p^4)\right], \tag{3.3}$$

$$\Gamma^{(4)}_{a_1a_2,b_1b_2,c_1c_2,d_1d_2}(0) = \left.\frac{\delta^4\Gamma}{\delta\Phi^\dagger_{a_1a_2}\delta\Phi^\dagger_{b_1b_2}\delta\Phi_{c_1c_2}\delta\Phi_{d_1d_2}}\right|_{\text{zero mom.}}$$
$$= Z_\phi^{-2} m^{4-d}\left(uU_{a_1a_2,b_1b_2,c_1c_2,d_1d_2} + vV_{a_1a_2,b_1b_2,c_1c_2,d_1d_2}\right) \tag{3.4}$$

where Γ is the generator of the one-particle irreducible correlation functions (effective action), and

$$U_{a_1a_2,b_1b_2,c_1c_2,d_1d_2} = \frac{1}{2}\left(\delta_{a_1c_1}\delta_{b_1d_1}\delta_{a_2c_2}\delta_{b_2d_2} + \delta_{a_1d_1}\delta_{b_1c_1}\delta_{a_2d_2}\delta_{b_2c_2}\right),$$
$$V_{a_1a_2,b_1b_2,c_1c_2,d_1d_2} = \frac{1}{2}\left(\delta_{a_1c_1}\delta_{b_1d_1}\delta_{a_2d_2}\delta_{b_2c_2} + \delta_{a_1d_1}\delta_{b_1c_1}\delta_{a_2c_2}\delta_{b_2d_2}\right).$$

The FP's of the theory are given by the common zeroes of the Callan-Symanzik β-functions

$$\beta_u(u,v) = m\left.\frac{\partial u}{\partial m}\right|_{u_0,v_0}, \qquad \beta_v(u,v) = m\left.\frac{\partial v}{\partial m}\right|_{u_0,v_0}. \tag{3.5}$$

Their stability is controlled by the eigenvalues of the matrix

$$\Omega = \begin{pmatrix} \partial\beta_u/\partial u & \partial\beta_u/\partial v \\ \partial\beta_v/\partial u & \partial\beta_v/\partial v \end{pmatrix}. \tag{3.6}$$

A FP is stable if all the eigenvalues of its stability matrix have positive real part. According to RG theory, a stable FP in the region $v > 0$ would imply the existence of a universality class describing continuous transitions characterized by the symmetry breaking (3.2). Therefore, in order to establish the existence of a 3-d universality class with symmetry-breaking pattern (3.2) and a complex $N \times N$ matrix order parameter, one must search for stable zeroes of the β-functions.

One can easily identify two FP's in the theory described by the lagrangian $\mathcal{L}_{\mathrm{U}(N)}$, without performing any calculation. The first one is the gaussian FP for $u = v = 0$, which is always unstable. Since for $v_0 = 0$ the lagrangian $\mathcal{L}_{\mathrm{U}(N)}$ becomes equivalent to the one of the $\mathrm{O}(2N^2)$-symmetric model, the corresponding $\mathrm{O}(2N^2)$ FP must exist in the $v = 0$ axis for $u > 0$. One-loop ϵ-expansion computations do not find other FP's [1]. The location of the $\mathrm{O}(2N^2)$ FP can be obtained by using known results for the $\mathrm{O}(M)$-symmetric theories. Estimates relevant for our study, i.e. for O(8), O(18), O(32) and O(72), corresponding to $N = 2, 3, 4, 6$, can be found in refs. [49, 50]. Moreover, using the results of refs. [11, 51] on the stability of the $\mathrm{O}(N)$-symmetric FP under generic perturbations, one can already conclude that the $\mathrm{O}(2N^2)$ FP is unstable for any $N \geq 2$. Indeed, the term $\mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2$ in the lagrangian $\mathcal{L}_{\mathrm{U}(N)}$ is a particular combination of quartic operators transforming as the spin-0 and spin-4 representations of the $\mathrm{O}(2N^2)$ group, and any spin-4 quartic perturbation is relevant at the $\mathrm{O}(M)$ FP for $M \geq 3$ [11, 51]. The RG dimension $y_{4,4}$ of the spin-4 quartic operators at the $\mathrm{O}(M)$ FP can be computed by using the six-loop FD results and the five-loop ϵ-expansion results for the related crossover exponent $\phi_{4,4} = y_{4,4}\nu$ reported in ref. [11]. We obtain the estimates $y_{4,4} = 0.386(7), 0.673(6), 0.808(5), 0.914(3)$ respectively for O(8), O(16), O(32), and O(72), that are the cases interesting for this work. Since $y_{4,4} > 0$, the $\mathrm{O}(2N^2)$ FP is unstable in all cases.

3.2 Fixed-dimension expansion in three dimensions

In order to investigate the existence of new FP's, we computed the FD expansion of the 3-d Callan-Symanzik β-functions (3.5) to six loops. This is rather cumbersome, since approximately one thousand Feynman diagrams must be evaluated. We employed a symbolic manipulation program, which generated the diagrams and computed the symmetry and group factors of each of them. We used the numerical results compiled in ref. [53] for the integrals associated with each diagram.

We report the series in terms of the rescaled couplings $\bar{u}$ and $\bar{v}$,

$$u \equiv \frac{16\pi}{4+N^2}\bar{u}\,, \qquad v \equiv \frac{16\pi}{4+N^2}\bar{v}\,. \tag{3.7}$$

The six-loop expansions of the β-functions corresponding to these rescaled variables are given by

$$\begin{aligned}\beta_{\bar{u}}(\bar{u},\bar{v}) &= -\bar{u}+\bar{u}^2+\frac{4N}{4+N^2}\bar{u}\bar{v}+\frac{3}{4+N^2}\bar{v}^2-\frac{190+82N^2}{27(4+N^2)^2}\bar{u}^3-\frac{400N}{27(4+N^2)^2}\bar{u}^2\bar{v}-\\ &\quad -\frac{370+46N^2}{27(4+N^2)^2}\bar{u}\bar{v}^2-\frac{4N}{(4+N^2)^2}\bar{v}^3+\sum_{i+j\geq 4} b^u_{ij}\frac{1}{(4+N^2)^{i+j}}\,\bar{u}^i\bar{v}^j\,,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\beta_{\bar{v}}(\bar{u},\bar{v}) &= -\bar{v}+\frac{6}{4+N^2}\bar{u}\bar{v}+\frac{2N}{4+N^2}\bar{v}^2-\frac{370+46N^2}{27(4+N^2)^2}\bar{u}^2\bar{v}-\\ &\quad -\frac{400N}{27(4+N^2)^2}\bar{u}\bar{v}^2-\frac{136+28N^2}{27(4+N^2)^2}\bar{v}^3+\sum_{i+j\geq 4} b^v_{ij}\frac{1}{(4+N^2)^{i+j}}\,\bar{u}^i\bar{v}^j\,.\end{aligned} \tag{3.9}$$

The coefficients b^u_{ij} and b^v_{ij}, with $4 \leq i+j \leq 7$, are reported in the tables 1 and 2, respectively. Actually, we also computed the RG functions η_ϕ and η_t, cf. eq. (2.10), to six loops. But, since we will not use them, we do not report their series. Of course, they are available on request.

We have done a number of checks by using known results available in the literature. For $\bar{v}=0$, the expansion of $\beta_{\bar{u}}$ in powers of $\bar{u}$ reproduces that of the β-function of the O$(2N^2)$-symmetric model that can be found in refs. [53, 49] to six loops. Since we computed the series for generic U$(N)_L \otimes$ U$(M)_R$ models, we also verified nontrivial relations holding for $M=1$ and any N, such as

$$\beta_u(z+y,z-y;N,M=1)+\beta_v(z+y,z-y;N,M=1)=\bar{\beta}(z;2N)\,, \tag{3.10}$$

where $\bar{\beta}(z;2N)$ is the β-function of the O$(2N)$ model and we are considering unrescaled couplings normalized as in eq. (2.7).

In order to resum the perturbative expansions, we use the conformal-mapping method described in section 2.3. For this purpose we compute the function A defined in eq. (2.20) by studying the saddle-point solutions (instantons) of the classical field equations. Under the assumption that the group and space structure of the saddle-point solutions decouple, the computation can be easily reduced to the case of the one-component Φ^4 theory described in detail in, e.g., refs. [33, 45]. Here we only report the results, without giving the details of the

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i,j	b^u_{ij}
4,0	$99.8202086 + 79.8954292\,N^2 + 16.4329798\,N^4 + 0.674471379\,N^6$
3,1	$329.227699\,N + 108.894438\,N^3 + 6.64687835\,N^5$
2,2	$366.245988 + 251.386542\,N^2 + 41.2000397\,N^4 + 0.310944602\,N^6$
1,3	$280.991858\,N + 83.3331935\,N^3 + 3.27130728\,N^5$
0,4	$65.0697173 + 40.0695380\,N^2 + 5.95052716\,N^4$
5,0	$-458.051683 - 415.773535\,N^2 - 111.135357\,N^4 - 8.64375912\,N^6 + 0.0778229492\,N^8$
4,1	$-2064.17070\,N - 880.190454\,N^3 - 85.6254168\,N^5 + 1.35288201\,N^7$
3,2	$-2647.21638 - 2601.32272\,N^2 - 492.107570\,N^4 - 1.80697817\,N^6$
2,3	$-3634.53973\,N - 1183.42261\,N^3 - 70.4182766\,N^5 - 0.430339129\,N^7$
1,4	$-1349.56809 - 1084.80505\,N^2 - 197.491458\,N^4 - 2.65955034\,N^6$
0,5	$-336.753422\,N - 103.463574\,N^3 - 4.81880464\,N^5$
6,0	$2596.27212 + 2603.95922\,N^2 + 822.999966\,N^4 + 86.8071683\,N^6 + 0.911941440\,N^8$ $+0.0256180937\,N^{10}$
5,1	$14967.9450\,N + 7570.43530\,N^3 + 1000.72434\,N^5 + 13.4051039\,N^7 + 0.625521452\,N^9$
4,2	$21496.0853 + 26926.0889\,N^2 + 6177.06793\,N^4 + 216.477923\,N^6 + 4.80379167\,N^8$
3,3	$44213.7572\,N + 17686.3589\,N^3 + 1767.24483\,N^5 + 27.2537306\,N^7$
2,4	$20802.2607 + 22953.0190\,N^2 + 5124.40912\,N^4 + 171.846652\,N^6 + 0.0681850017\,N^8$
1,5	$12267.0624\,N + 4868.45580\,N^3 + 459.187339\,N^5 + 2.19119772\,N^7$
0,6	$1359.90833 + 1466.49761\,N^2 + 303.552043\,N^4 + 5.48047785\,N^6$
7,0	$-16956.2524 - 18511.7426\,N^2 - 6735.47522\,N^4 - 924.744197\,N^6 - 32.1576679\,N^8$ $+0.314797775\,N^{10} + 0.0117120916\,N^{12}$
6,1	$-119503.736\,N - 69794.4619\,N^3 - 11896.6226\,N^5 - 458.710404\,N^7 + 6.58801236\,N^9$ $+0.363425149\,N^{11}$
5,2	$-188784.360 - 284537.687\,N^2 - 78459.3804\,N^4 - 4739.14953\,N^6 + 22.5892998\,N^8$ $+3.03196500\,N^{10}$
4,3	$-528347.831\,N - 257216.321\,N^3 - 32795.4801\,N^5 - 331.733950\,N^7 + 11.6377106\,N^9$
3,4	$-292480.092 - 416299.806\,N^2 - 103998.225\,N^4 - 4515.48789\,N^6 + 8.83297998\,N^8$
2,5	$-298671.340\,N - 136395.073\,N^3 - 16227.9484\,N^5 - 201.034856\,N^7 - 0.500032757\,N^9$
1,6	$-59511.6725 - 77035.0275\,N^2 - 18555.3684\,N^4 - 766.457739\,N^6 - 3.10874638\,N^8$
0,7	$-13984.0313\,N - 6024.80289\,N^3 - 653.107647\,N^5 - 5.22722079\,N^7$

Table 1: The coefficients b^u_{ij}, cf. eq. (3.8).

calculations which are rather standard. Consider a generic quantity $S(\bar u,\bar v) = \sum_{ij} c_{ij}\bar u^i \bar v^j$ and the corresponding expansion

$$S(g\bar u, g\bar v) = \sum_n s_n(\bar u,\bar v) g^n \,. \qquad (3.11)$$

The large-order behavior is given by

$$s_n \sim n![-A(\bar u,\bar v)]^n\, n^b \,, \qquad (3.12)$$

$$A(\bar u,\bar v) = A_N \mathrm{Max}\left[\bar u + \bar v,\, \bar u + \frac{\bar v}{N}\right], \qquad (3.13)$$

$$A_N = \frac{24\pi}{4+N^2} a \,, \qquad (3.14)$$

where $a \approx 0.00881962$. The expansion is Borel summable in the region

$$\bar u + \frac{\bar v}{N} \geq 0 \,, \qquad \bar u + \bar v \geq 0 \,, \qquad (3.15)$$

which, as already mentioned in section 2.3, is related to the stability region of the quartic potential in the lagrangian $\mathcal{L}_{\mathrm{U}(N)}$, which is given by $u_0 + v_0/N \geq 0$ and $u_0 + v_0 \geq 0$ [1].

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i,j	b^v_{ij}
4,0	0
3,1	$234.666985 + 100.520648\, N^2 + 7.96126092\, N^4 - 0.625553655\, N^6$
2,2	$426.449736\, N + 97.5987129\, N^3 - 2.25343030\, N^5$
1,3	$254.535978 + 144.854814\, N^2 + 20.3052050\, N^4$
0,4	$66.3459284\, N + 18.8271989\, N^3 + 0.560179202\, N^5$
5,0	0
4,1	$-1258.17306 - 606.687101\, N^2 - 72.7688516\, N^4 - 1.08252821\, N^6 - 0.287326259\, N^8$
3,2	$-3271.88287\, N - 894.130891\, N^3 - 25.2461704\, N^5 - 1.55153180\, N^7$
2,3	$-2559.77598 - 2108.80199\, N^2 - 375.743846\, N^4 - 2.13233677\, N^6$
1,4	$-1708.58378\, N - 585.048888\, N^3 - 39.4757354\, N^5$
0,5	$-266.102670 - 237.227857\, N^2 - 44.6632567\, N^4 - 0.496927343\, N^6$
6,0	0
5,1	$8093.66025 + 4354.56520\, N^2 + 652.419979\, N^4 + 13.7782897\, N^6 - 1.54366403\, N^8$ $-0.159052164\, N^{10}$
4,2	$27834.1272\, N + 8658.04010\, N^3 + 371.284973\, N^5 - 18.1481024\, N^7 - 1.18751905\, N^9$
3,3	$26250.5520 + 27345.7269\, N^2 + 4995.12745\, N^4 - 63.2324448\, N^6 - 3.26781317\, N^8$
2,4	$30725.1752\, N + 11079.3449\, N^3 + 816.715993\, N^5 - 8.19919537\, N^7$
1,5	$8470.20827 + 8744.03485\, N^2 + 1815.09720\, N^4 + 39.6191261\, N^6$
0,6	$2108.57715\, N + 797.860452\, N^3 + 68.8157914\, N^5 + 0.284187609\, N^7$
7,0	0
6,1	$-58941.8989 - 34933.0997\, N^2 - 6302.78433\, N^4 - 295.542877\, N^6 + 0.957520387\, N^8$ $-1.27213433\, N^{10} - 0.0997279331\, N^{12}$
5,2	$-254016.821\, N - 89637.8297\, N^3 - 6227.84400\, N^5 + 32.2770587\, N^7 - 14.8328121\, N^9$ $-0.951111299\, N^{11}$
4,3	$-275746.727 - 345193.386\, N^2 - 70372.4321\, N^4 - 538.101215\, N^6 - 67.5278644\, N^8$ $-3.69213913\, N^{10}$
3,4	$-476545.568\, N - 197140.916\, N^3 - 19569.5278\, N^5 - 52.0633373\, N^7 - 8.74103639\, N^9$
2,5	$-174726.233 - 224936.606\, N^2 - 53193.0192\, N^4 - 2001.32523\, N^6 - 7.87773734\, N^8$
1,6	$-96977.9325\, N - 44001.9263\, N^3 - 5202.93250\, N^5 - 65.8929276\, N^7$
0,7	$-8282.26037 - 11331.8800\, N^2 - 2764.77952\, N^4 - 113.637842\, N^6 - 0.318785501\, N^8$

Table 2: The coefficients b^v_{ij}, cf. eq. (3.9).

3.3 Analysis of the series and results

The knowledge of the large-order behavior of the series allows us to employ the conformal-mapping resummation method. As already mentioned in section 2.3, this is essentially an extension of the method already applied in refs. [41, 42] to resum the perturbative expansions in $O(N)$ theories. Explicitly, given a series

$$R(\bar{u},\bar{v}) = \sum_{hk} R_{hk} \bar{u}^h \bar{v}^k \,, \tag{3.16}$$

we rewrite it as

$$R(\bar{u},\bar{v}) = R(g\bar{u}, g\bar{v})\big|_{g=1} = \hat{R}(\bar{u},\bar{v}; g=1)\,, \tag{3.17}$$

where

$$\hat{R}(\bar{u},\bar{v};g) = \sum_n r_n(\bar{u},\bar{v}) g^n \,, \qquad r_n(\bar{u},\bar{v}) = \sum_{h+k=n} R_{hk} \bar{u}^h \bar{v}^k \,, \tag{3.18}$$

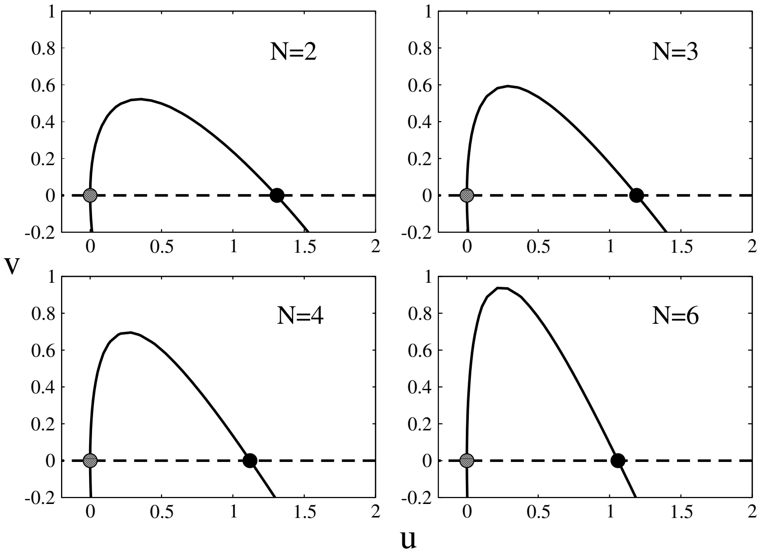

Figure 1: Zeroes of the β-functions for $N = 2, 3, 4, 6$ (solid line for β_u, dashed line for β_v). The grey and black blobs represent the gaussian and the $\mathrm{O}(2N^2)$ FP's, respectively.

Then, we consider approximants $E(R)_p(\bar{u}, \bar{v}; b, \alpha; g)$ defined by

$$E(R)_p(\bar{u}, \bar{v}; b, \alpha; g) = \sum_{k=0}^{p} B_k(\bar{u}, \bar{v}; b, \alpha) \int_0^\infty dt\, t^b e^{-t} \frac{Y(gt; \bar{u}, \bar{v})^k}{[1 - Y(gt; \bar{u}, \bar{v})]^\alpha}\,, \tag{3.19}$$

where

$$Y(x; y, z) = \frac{\sqrt{1 + xA(y, z)} - 1}{\sqrt{1 + xA(y, z)} + 1}\,, \tag{3.20}$$

and $A(y, z)$ is given in eq. (3.13). The coefficients $B_k(\bar{u}, \bar{v}; b, \alpha)$ are determined by the requirement that the expansion of $E(R)_p(\bar{u}, \bar{v}; b, \alpha; g)$ in powers of g gives $\hat{R}(\bar{u}, \bar{v}; g)$ to order p. For each value of b, α, and p, $E(R)_p(\bar{u}, \bar{v}; b, \alpha; g = 1)$ provides an estimate of $R(\bar{u}, \bar{v})$. The parameters b and α are arbitrary and can be used to optimize the resummation, see, e.g., refs. [41, 42, 48]. For example, one may require that the results are least dependent on the number p of terms. Usually one considers values in the region $0 \leq b \lesssim 15$ and $-1 \leq \alpha \lesssim 5$. The dependence of the results with respect to variations of b and α around their optimal values provides indications of the uncertainty.

We have applied the conformal-mapping resummation method to the six-loop series of the β-functions, and then determined their zeroes. The zeroes of the $\beta_{\bar{u}}$ and $\beta_{\bar{v}}$ are shown in figure 1 for the cases $N = 2, 3, 4, 6$. In particular, $\beta_{\bar{v}}(\bar{u}, \bar{v})$ turns out to vanish only along the $\bar{v} = 0$ axis for any N. The figures show that no other FP exists beside the gaussian and the $\mathrm{O}(2N^2)$ FP's located along the $\bar{v} = 0$ axis, and already predicted by general considerations, see the discussion at the end of section 3.1. The $\bar{u}$-coordinate of the $\mathrm{O}(2N^2)$ FP is $\bar{u}^*_{O(M)} \approx 1.30, 1.19, 1.12, 1.06$ respectively for $N = 2, 3, 4, 6$ (these estimates are in agreement with those that can be found in refs. [49, 50] for the position of the FP in $\mathrm{O}(M)$-symmetric theories). The curves appearing in figure 1 are quite robust with respect to the order of the series and to variations of the resummation parameters. None of the approximants of the series for $\beta_{\bar{v}}$, from four to six loops, shows zeroes for

$|\bar{v}| > 0$ in the region in which the resummation is still reliable, i.e. for $-2 \lesssim \bar{u}, \bar{v} \lesssim 4$, which is quite a large region if compared with the location of the $\mathrm{O}(2N^2)$ FP along the $\bar{v} = 0$ axis. We recall that the relevant region is the one with $v_0 > 0$ that corresponds to $\bar{v} > 0$. Concerning the zeroes of $\beta_{\bar{u}}(\bar{u}, \bar{v})$, all zeroes obtained by changing the order (from four to six loops) and by varying b and α in the above-mentioned range, lie within the width of the full lines. Moreover, perfectly consistent results have also been obtained by employing the Padé-Borel method, which assumes only Borel summability, without using the information on the position of the Borel-transform singularities. The existence of FP's outside the region where the resummation is reliable is quite unlikely.

In conclusion, we do not find evidence of new FP's. According to RG theory, this means that a consistent model for a three-dimensional continuous transition characterized by the symmetry-breaking pattern $[\mathrm{U}(N_f)_L \otimes \mathrm{U}(N_f)_R]/\,\mathrm{U}(1)_V \to \mathrm{U}(N_f)_V/\,\mathrm{U}(1)_V$ does not exist for any $N \geq 2$. Therefore, the phase transition in such systems must be a first-order one. Our results confirm and put on firmer ground earlier claims [1] based on a first-order perturbative calculation within the ϵ-expansion scheme.

4. The $\mathrm{SU}(N)_L \otimes \mathrm{SU}(N)_R$ effective theory

4.1 General considerations

If the $\mathrm{U}(1)_A$ symmetry is explicitly broken by the anomaly, the axial $\mathrm{U}(1)_A$ symmetry is reduced to $\mathbb{Z}(N)_A$ [1]. If we rewrite the symmetry-breaking pattern (3.2) in the form $\mathrm{U}(1)_A \otimes [\mathrm{SU}(N)/\mathbb{Z}(N)]_L \otimes [\mathrm{SU}(N)/\mathbb{Z}(N)]_R \to \mathrm{SU}(N)_V/\mathbb{Z}(N)_V$ and replace $\mathrm{U}(1)_A$ with $\mathbb{Z}(N)_A$, we obtain

$$[\mathrm{SU}(N)_L \otimes \mathrm{SU}(N)_R]/\mathbb{Z}(N)_V \to \mathrm{SU}(N)_V/\mathbb{Z}(N)_V\,. \tag{4.1}$$

The LGW theory with a complex $N \times N$ matrix order parameter and such a breaking pattern is obtained by adding new terms to the lagrangian $\mathcal{L}_{\mathrm{U}(N)}$. The most relevant one is proportional to the determinant of Φ and thus we obtain the effective three-dimensional theory

$$\mathcal{L}_{\mathrm{SU}(N)} = \mathcal{L}_{\mathrm{U}(N)} + w_0 \left(\det \Phi^\dagger + \det \Phi\right). \tag{4.2}$$

Here w_0 is related to the breaking of the $\mathrm{U}(1)_A$ symmetry.

For $N = 1$ the added term in the lagrangian (4.2) is equivalent to an external field coupled to the order parameter. Therefore, it smooths out the continuous XY transition realized in its absence, leaving us with an analytic crossover.

For $N = 2$ the phase diagram is more complex and is discussed in detail in the appendix. In this case, the effective lagrangian containing all possible interactions with at most four fields is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{SU}(2)} = {} & \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r\,\mathrm{Tr}\,\Phi^\dagger\Phi + \\ & + w_0 \left(\det \Phi^\dagger + \det \Phi\right) + \frac{u_0}{4}\left(\mathrm{Tr}\,\Phi^\dagger\Phi\right)^2 + \frac{v_0}{4}\,\mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2 + \\ & + \frac{x_0}{4}\left(\mathrm{Tr}\,\Phi^\dagger\Phi\right)\left(\det\Phi^\dagger + \det\Phi\right) + \frac{y_0}{4}\left[(\det\Phi^\dagger)^2 + (\det\Phi)^2\right],\end{aligned} \tag{4.3}$$

where Φ_{ij} is a complex $2{\times}2$ matrix.

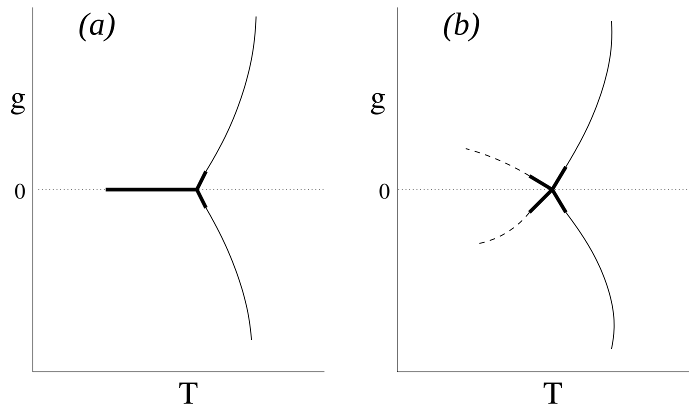

Figure 2: Sketches of possible phase diagrams with bicritical and tetracritical points, case (a) and (b) respectively. Full lines correspond to continuous transitions with symmetry-breaking pattern $\mathrm{SO}(4) \to \mathrm{SO}(3)$, which are therefore the relevant ones for QCD. Dashed lines correspond to Ising transitions, not realized in QCD. Thick lines indicate first-order transitions.

Since the lagrangian (4.3) has two mass terms, one expects several transition lines with a multicritical point. In the case relevant for QCD the multicritical point should be identified with the $\mathrm{U}(2) \otimes \mathrm{U}(2)$ theory, and therefore it should correspond to $w_0 = 0$, $x_0 = 0$, $y_0 = 0$. As we discussed in the preceding section, such a multicritical point corresponds to a first-order transition. Since a first-order transition is generally robust with respect to perturbations, it should persist even for a small breaking of the $\mathrm{U}(1)_A$ symmetry. Therefore, on the basis of the analysis presented in the appendix we expect two possible phase diagrams as a function of the temperature T and of the $\mathrm{U}(1)_A$ breaking g, see figure 2. In the bicritical case, see figure 2a, the continuous phase transitions are associated with the symmetry-breaking pattern (4.1) which is equivalent to $\mathrm{SO}(4) \to \mathrm{SO}(3)$ [1]. Therefore, according to universality arguments, if two-flavor QCD undergoes a continuous transition, its critical behavior at T_c should be that of the 3-d $\mathrm{SO}(4)$ universality class [1, 2, 3], which has been accurately studied in the literature, see, e.g., the recent refs. [34, 42, 54, 55, 56] and references therein. In the tetracritical case, see figure 2b, only two continuous transition lines are compatible with the symmetry-breaking pattern expected in QCD, those indicated with the full lines, see the discussion in the appendix. Note that the second-order lines end at particular values $g_{\rm tr}$ which are tricritical points characterized by mean-field behavior with logarithmic corrections. In conclusion, the transition may be of first-order if the $\mathrm{U}(1)_A$ breaking is small, second-order in the opposite case, and may be of mean-field type for a very specific value of g.[7] Of course, a specific study of QCD is needed to identify which case is effectively realized. Lattice simulations of two-flavor QCD favor a continuous transition, see, e.g., refs. [23]–[27], with scaling properties that are substantially consistent with the 3-d $\mathrm{SO}(4)$ universality class. In the large-N_c limit, N_c being the number of colors, the anomaly effects are suppressed by powers of $1/N_c$. Therefore, the finite-temperature

[7] The effective $\mathrm{U}(1)_A$-symmetry breaking at finite temperature has been investigated on the lattice. The $\mathrm{U}(1)_A$ symmetry appears not to be restored at T_c. However, the effective breaking of the axial $\mathrm{U}(1)_A$ symmetry appears substantially reduced especially above T_c, as inferred from the difference between the correlators in the pion and δ channels. See, e.g., refs. [29, 30, 31, 57].

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transition is expected to be of first order. This picture is also supported by the fact that the finite-temperature transition in pure $\mathrm{SU}(N_c)$ gauge theories appears to be first order for sufficiently large values of N_c [58], i.e. $N_c \geq 3$, and the quark contributions are expected to be suppressed by a factor $1/N_c$ for large N_c. This considerations suggest that for $N_c \geq N_{\min}$ — Monte Carlo simulations indicate $N_{\min} > 3$ — the finite-temperature transition with two flavors changes its nature from continuous to first order.

For $N = 3$, the determinant is cubic in the field Φ. Usually, a cubic term drives to a first-order transition, and therefore the addition of the determinant is not expected to soften the original first-order transition to a continuous one. Lattice simulations of QCD confirm this picture, see, e.g., refs. [18, 19, 20, 21, 22].

For $N = 4$ the determinant is a quartic-order term, giving rise to a generalized LGW Φ^4 theory with three quartic parameters. The effects of the determinant in this case will be discussed more carefully in section 4.2. As we shall see, the determinant does not give rise to new FP's, and therefore the phase transition remains first order also in the presence of a substantial effective breaking of the $\mathrm{U}(1)_A$ symmetry.

Finally, for $N \geq 5$ the determinant term is irrelevant because it gives rise to polynomial terms of degree higher than four. Therefore, for $N \geq 5$ the nature of the transition should not change. The lattice results for $N = 6$ of ref. [19] are consistent with this picture.

4.2 Fixed dimension expansion in the case $N = 4$

As argued above, the theory for $N = 4$ is the only case in which it is necessary to investigate the effect of the determinant, since no arguments exist against the possibility that such a term softens the first-order transition expected in its absence. The three-dimensional RG flow of the lagrangian $\mathcal{L}_{\mathrm{SU}(N)}$ for $N = 4$, cf. eq. (4.2), can be studied by a straightforward extension of the calculations done for the lagrangian $\mathcal{L}_{\mathrm{U}(N)}$.

Beside the renormalization conditions (3.3) and (3.4), we add a further relation

$$\begin{aligned}\hat{\Gamma}^{(4)}_{a_1a_2,b_1b_2,c_1c_2,d_1d_2}(0) &= \left.\frac{\delta^4\Gamma}{\delta\Phi_{a_1a_2}\delta\Phi_{b_1b_2}\delta\Phi_{c_1c_2}\delta\Phi_{d_1d_2}}\right|_{\text{zero mom.}} \\ &= Z_\phi^{-2} m^{4-d} w \epsilon_{a_1b_1c_1d_1}\epsilon_{a_2b_2c_2d_2}\,, \end{aligned} \tag{4.4}$$

where ϵ_{ijkl} is the completely antisymmetric tensor ($\epsilon_{1234} = 1$).

We computed the perturbative expansion of correlation functions $\Gamma^{(2)}$, $\Gamma^{(4)}$ and $\hat{\Gamma}^{(4)}$ to six loops. Beside the rescaled variables $\bar{u}$ and $\bar{v}$, cf. eq. (3.7) with $N = 4$, we introduce $\bar{w}$ defined by $w = 4\pi\bar{w}/5$. The corresponding β-functions are given by

$$\begin{aligned}\beta_{\bar{u}}(\bar{u},\bar{v},\bar{w}) &= -\bar{u} + \bar{u}^2 + \frac{4}{5}\bar{u}\bar{v} + \frac{3}{20}\bar{v}^2 + \frac{2}{5}\bar{w}^2 - \frac{751}{5400}\bar{u}^3 - \frac{4}{27}\bar{u}^2\bar{v} - \\ &\quad - \frac{553}{5400}\bar{u}\bar{v}^2 - \frac{41}{225}\bar{u}\bar{w}^2 - \frac{1}{25}\bar{v}^3 - \frac{2}{75}\bar{v}\bar{w}^2 + \sum_{i+j+k\geq 4} b^u_{ijk}\,\bar{u}^i\bar{v}^j\bar{w}^k\,, \end{aligned} \tag{4.5}$$

$$\begin{aligned}\beta_{\bar{v}}(\bar{u},\bar{v},\bar{w}) &= -\bar{v} + \frac{3}{10}\bar{u}\bar{v} + \frac{2}{5}\bar{v}^2 - \frac{2}{5}\bar{w}^2 - \frac{553}{5400}\bar{u}^2\bar{v} - \frac{4}{27}\bar{u}\bar{v}^2 + \\ &\quad + \frac{2}{25}\bar{u}\bar{w}^2 - \frac{73}{1350}\bar{v}^3 + \frac{2}{45}\bar{v}\bar{w}^2 + \sum_{i+j+k\geq 4} b^v_{ijk}\,\bar{u}^i\bar{v}^j\bar{w}^k\,, \end{aligned} \tag{4.6}$$

$$\beta_{\bar w}(\bar u,\bar v,\bar w) = -\bar w + \frac{3}{10}\bar u\bar w - \frac{3}{10}\bar v\bar w - \frac{553}{5400}\bar u^2\bar w - \\ -\frac{11}{1350}\bar u\bar v\bar w + \frac{61}{2700}\bar v^2\bar w + \frac{4}{225}\bar w^3 + \sum_{i+j+k\geq 4} b^w_{ijk}\,\bar u^i\bar v^j\bar w^k\,. \tag{4.7}$$

The coefficients b^u_{ijk}, b^v_{ijk}, and b^w_{ijk}, with $4 \leq i+j+k \leq 7$, are reported in the table 3. Note that $\beta_{\bar u}$ and $\beta_{\bar v}$ are even functions of $\bar w$, while $\beta_{\bar w}$ is an odd function of $\bar w$.

We have also determined the large-order behavior of the series. Writing

$$S(g\bar u, g\bar v, g\bar w) = \sum_n s_n(\bar u,\bar v,\bar w)g^n\,, \tag{4.8}$$

we found

$$s_n \sim n![-A(\bar u,\bar v,\bar w)]^n\, n^b\,, \tag{4.9}$$
$$A(\bar u,\bar v,\bar w) = \frac{6\pi a}{5}\,\mathrm{Max}\left[\bar u+\bar v,\;\bar u+\frac{\bar v}{4}+\frac{|\bar w|}{2}\right].$$

The series turn out to be Borel summable for

$$\bar u+\bar v \geq 0\,, \qquad \bar u+\frac{\bar v}{4}-\frac{|\bar w|}{2} \geq 0\,, \tag{4.10}$$

which is again the stability region of the quartic potential (with respect to the bare couplings u_0, v_0, and w_0).

In order to resum the series of the β-functions and search for their zeroes, we applied the same method employed in section 3.3. The analysis of the series clearly shows that the new β-function $\beta_{\bar w}$ vanishes only for $\bar w = 0$. As a consequence, the FP's can only lie in the plane $\bar w = 0$, and therefore they are those of the $\mathrm{U}(4)_L\otimes\mathrm{U}(4)_R$ theory, see section 3.1, which are unstable. Since no stable FP is found, the phase transition is expected to be first order for any value of w_0, and therefore also in the presence of a substantial breaking of the $\mathrm{U}(1)_A$ symmetry.

Acknowledgments

We thank Pasquale Calabrese and Luigi Del Debbio for useful and interesting discussions.

A. The phase diagram of the $\mathrm{SU}(2)_L\otimes\mathrm{SU}(2)_R$ Φ^4 theory

In this appendix we discuss in more detail the phase diagram of the theory with lagrangian (4.3). It is easy to map $\mathcal{L}_{\mathrm{SU}(2)}$ into a new one, with two 4-dimensional vector fields and SO(4) symmetry. If

$$\Phi = \alpha + i\beta + i(\vec A + i\vec B)\cdot\vec\sigma\,, \tag{A.1}$$

where α, β, $\vec A$, and $\vec B$ are real and $\vec\sigma$ are the Pauli matrices, we define two fields

$$\phi_1 = 2(\alpha,\vec A)\,, \qquad \phi_2 = 2(\beta,\vec B)\,. \tag{A.2}$$

i,j,k	b^u_{ijk}	b^v_{ijk}	i,j,k	b^w_{ijk}
4,0,0	0.0521726542	0		
3,1,0	0.0943284891	0.0082425774	3,0,1	0.0082425774
2,2,0	0.101307937	0.0352787747	2,1,1	−0.0286626158
2,0,2	0.10529853	−0.0365379374		
1,3,0	0.0612944404	0.0485646592	1,2,1	−0.018209011
1,1,2	0.0096560684	−0.00856024106	1,0,3	0.0221761335
0,4,0	0.013934483	0.0127746747	0,3,1	−0.000434920883
0,2,2	−0.033458354	0.016749908	0,1,3	0.0230423535
0,0,4	−0.0158081156	0.00800526372		
5,0,0	−0.0205830351	0		
4,1,0	−0.0406573999	−0.0165182006	4,0,1	−0.0165182006
3,2,0	−0.0555154064	−0.0379950888	3,1,1	−0.0193283018
3,0,2	−0.0554550413	0.0106667354		
2,3,0	−0.0529488118	−0.0441328387	2,2,1	−0.01454706
2,1,2	−0.0101562476	−0.00163844246	2,0,3	−0.0155117024
1,4,0	−0.0250493063	−0.0264689428	1,3,1	−0.00480590003
1,2,2	0.0285562806	−0.00933100358	1,1,3	−0.00990890441
1,0,4	0.00460313911	0.00264466761		
0,5,0	−0.00403223074	−0.0054784239	0,4,1	−0.000312387727
0,3,2	0.00827027644	−0.00421798495	0,2,3	0.00104252159
0,1,4	−0.00074739545	0.000751462171	0,0,5	0.000775960184
6,0,0	0.0108927701	0		
5,1,0	0.0305113637	0.000519972577	5,0,1	0.000519972577
4,2,0	0.05054934	0.00682824039	4,1,1	−0.00833012972
4,0,2	0.0359528478	−0.00866192578		
3,3,0	0.055702591	0.0198339892	3,2,1	−0.0106000754
3,1,2	0.019505026	−0.00500471528	3,0,3	0.0107231224
2,4,0	0.0376289338	0.0239681302	2,3,1	−0.00602064077
2,2,2	−0.00918437911	−0.00049757924	2,1,3	0.00390222416
2,0,4	0.00649735532	−0.00478632298		
1,5,0	0.0135430912	0.0121143686	1,4,1	−0.00229637061
1,3,2	−0.00520588781	−0.00015566449	1,2,3	−0.00572268276
1,1,4	0.00892272331	−0.00644936705	1,0,5	−0.00169793557
0,6,0	0.00195283172	0.00210345121	0,5,1	−0.000366214748
0,4,2	−0.000042403606	0.0000232000102	0,3,3	−0.00241486655
0,2,4	0.00248127669	−0.000949407795	0,1,5	−0.000885328883
0,0,6	−0.0000138859841	0.000230911869		
7,0,0	−0.00578599786	0		
6,1,0	−0.0167118671	−0.00498926754	6,0,1	−0.00498926754
5,2,0	−0.0309210056	−0.0159991843	5,1,1	−0.0108285244
5,0,2	−0.0223201773	0.00295466283		
4,3,0	−0.0426110785	−0.0268087818	4,2,1	−0.0134843143
4,1,2	−0.0163196391	−0.00135070461	4,0,3	−0.00802619344
3,4,0	−0.0402292054	−0.029458448	3,3,1	−0.00957047769
3,2,2	0.00311298454	−0.00383770188	3,1,3	−0.00427874015
3,0,4	−0.0115432383	0.00461784073		
2,5,0	−0.0234111131	−0.0203943972	2,4,1	−0.00339634035
2,3,2	0.00452594729	−0.00157893132	2,2,3	0.00207216147
2,1,4	−0.0109265899	0.00435906825	2,0,5	−0.000429888812
1,6,0	−0.00733233759	−0.00750892782	1,5,1	−0.000558386825
1,4,2	0.00109452357	−0.000546285499	1,3,3	0.000859221454
1,2,4	−0.00117934744	0.000393239856	1,1,5	−0.00118266259
1,0,6	0.00103714759	−0.000591465198		
0,7,0	−0.000934334786	−0.00108103783	0,6,1	−0.000033642391
0,5,2	0.000134905627	−0.000206454834	0,4,3	−0.000144490092
0,3,4	0.0006868813	−0.000537767236	0,2,5	−0.000337875244
0,1,6	0.000550816481	−0.00046918919	0,0,7	−0.0000195695397

Table 3: The coefficients b^u_{ijk}, b^v_{ijk}, and b^w_{ijk}, cf. eqs. (4.5), (4.6) and (4.7). Those corresponding to values of i, j, k that are not reported are zero.

Then eq. (4.3) can be rewritten as

$$\mathcal{L}_{\mathrm{SU}(2)} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + \frac{r_1}{2}\phi_1^2 + \frac{r_2}{2}\phi_2^2 + \\ + \frac{a_0}{4!}(\phi_1^2)^2 + \frac{b_0}{4!}(\phi_2^2)^2 + \frac{c_0}{4!}\phi_1^2\phi_2^2 + \frac{d_0}{4!}(\phi_1 \cdot \phi_2)^2 , \qquad (A.3)$$

with

$$\begin{aligned} r_1 &= r + w_0 , \\ r_2 &= r - w_0 , \\ a_0 &= \frac{3}{4}(2u_0 + v_0 + 2x_0 + y_0) , \\ b_0 &= \frac{3}{4}(2u_0 + v_0 - 2x_0 + y_0) , \\ c_0 &= \frac{3}{2}(2u_0 + 3v_0 - y_0) , \\ d_0 &= -3(v_0 + y_0) . \end{aligned} \qquad (A.4)$$

The lagrangian (A.3) is stable for $a_0 > 0$, $b_0 > 0$, $c_0 > -2\sqrt{a_0 b_0}$, and $c_0 + d_0 > -2\sqrt{a_0 b_0}$. It is easy to identify the symmetries of the lagrangian (A.3). It is invariant under O(4) rotations of both fields ϕ_1 and ϕ_2 and under the $\mathbb{Z}(2)$ transformations in which $\phi_1 \to -\phi_1$ and $\phi_2 \to -\phi_2$ independently (if the two transformations are applied simultaneously, we obtain an O(4) transformation). Therefore, the symmetry group is $\mathbb{Z}(2) \otimes O(4)$. It is interesting to understand these symmetries in the context of the original model (4.3). First, there is the invariance under $\Phi \to A\Phi B$, A, $B \in \mathrm{SU}(2)$ that corresponds to the symmetry $[\mathrm{SU}(2) \otimes \mathrm{SU}(2)]/\mathbb{Z}(2)$, which is equivalent to SO(4). The other two $\mathbb{Z}(2)$ symmetries correspond to the transformations $\Phi \to \Phi^\dagger$ and $\Phi \to \Phi^T$. Note that these two transformations are symmetries of the effective theory for any N. We have not mentioned them before because they do not play any role at the QCD transition.

The lagrangian (A.3) has two independent quadratic (mass) terms and thus it describes critical and multicritical transitions. It is easy to determine the phase diagram of the theory in the mean-field approximation, extending the analysis of Liu and Fisher [59]. In this case, one assumes that the basic fields are space-independent, thereby neglecting the effect of fluctuations. In this approximations two different phase diagrams are found, see figure 3. In one case, there are three phases: 1) a disordered one for $r_1 > 0$ and $r_2 > 0$; 2) an ordered phase in which $\phi_1 \neq 0$ and $\phi_2 = 0$, with $r_1 < 0$ and $r_2 > \sqrt{b_0/a_0} r_1$; 3) an ordered phase in which $\phi_1 = 0$ and $\phi_2 \neq 0$, with $r_2 < 0$ and $r_2 < \sqrt{b_0/a_0} r_1$. The multicritical point corresponds to $r_1 = r_2 = 0$, and, according to standard terminology, is named bicritical in this case. The phase boundaries between the disordered phase and the ordered ones correspond to second-order transitions with symmetry breaking $O(4) \to O(3)$, while the boundary $r_2 = \sqrt{b_0/a_0} r_1$, $r_1 < 0$, $r_2 < 0$, corresponds to a first-order transition. In the second case, there are four phases: 1) a disordered one for $r_1 > 0$ and $r_2 > 0$; 2) an ordered phase in which $\phi_1 \neq 0$ and $\phi_2 = 0$, with $r_1 < 0$, bounded by a line depending on the quartic parameters; 3) an ordered phase in which $\phi_1 = 0$ and $\phi_2 \neq 0$, with $r_2 < 0$, bounded by a line depending on the quartic parameters; 4) an ordered phase in which both $\phi_1 \neq 0$ and

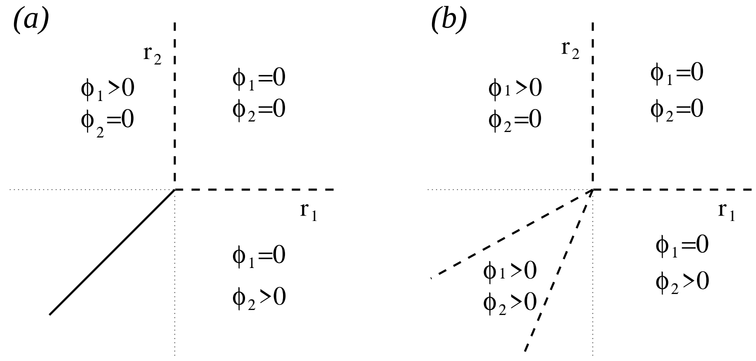

Figure 3: Sketches of the possible phase diagrams in the mean-field approximation: case (a) is characterized by a bicritical point, case (b) by a tetracritical point. The dashed and thick lines represent second- and first-order transitions, respectively.

$\phi_2 \neq 0$, bounded by the two lines considered above. The multicritical point corresponds to $r_1 = r_2 = 0$, and is called tetracritical. The phase boundaries always correspond to second-order transitions.

The nature of the phase diagram depends on the values of the quartic parameters. The multicritical point is bicritical for $c_0 > 2\sqrt{a_0 b_0}$ and $c_0 + d_0 > 2\sqrt{a_0 b_0}$. In all other cases it is tetracritical. In the tetracritical case, it is of interest to understand the symmetry breaking observed at each transition. The transitions between the disordered phase and the ordered ones always correspond to the breaking $O(4) \to O(3)$, or more precisely to $\mathbb{Z}(2) \otimes O(4) \to \mathbb{Z}(2) \otimes O(3)$. The additional $\mathbb{Z}(2)$ corresponds to reflections of the nonmagnetized field with respect to the plane orthogonal to the magnetized one: for instance, if ϕ_1 is magnetized in the first direction, then the symmetry $(\phi_2)_1 \to -(\phi_2)_1$, $(\phi_2)_i \to (\phi_2)_i$ $(i \geq 2)$ is not broken. In the presence of fluctuations these transitions are expected to belong to the O(4) universality class. The transition lines bounding region 4 may correspond to different symmetry-breaking patterns. If $d_0 < 0$ and $-2\sqrt{a_0 b_0} < c_0 + d_0 < 2\sqrt{a_0 b_0}$, in phase 4 ϕ_1 and ϕ_2 are parallel. At the boundaries the $\mathbb{Z}(2)$-symmetry we discussed above is broken and the symmetry gets reduced from $\mathbb{Z}(2) \otimes O(3)$ to $O(3)$. Therefore, these lines should correspond to Ising transitions. On the other hand, if $d_0 > 0$ and $-2\sqrt{a_0 b_0} < c_0 < 2\sqrt{a_0 b_0}$ in phase 4 ϕ_1 and ϕ_2 are orthogonal. Therefore, at the transition one observes the symmetry-breaking pattern $O(3) \to O(2)$ (more precisely $\mathbb{Z}(2) \otimes O(3) \to \mathbb{Z}(2) \otimes O(2)$). In this case, the critical behavior should belong to the O(3) universality class.

Finally, let us discuss the phase diagram predicted by mean-field theory in the limit in which the effective breaking of $\mathrm{U}(1)_A$ is small. If g parametrizes this breaking, then $c_0 = 3u_0 + \frac{9}{2}v_0 + O(g)$, $d_0 = -3v_0 + O(g)$, and $\sqrt{a_0 b_0} = \frac{3}{2}(u_0 + v_0/2) + O(g)$, so that $c_0 + d_0 = 2\sqrt{a_0 b_0} + O(g)$. Using the results reported above, we obtain that for $v_0 > 0$, i.e. $d_0 < 0$, the phase diagram is bicritical or tetracritical depending on the sign of the corrections proportional to g. Instead, for $v_0 < 0$, i.e. $d_0 > 0$, the phase diagram is tetracritical. We recall that the case relevant for QCD should be that with $v_0 > 0$, because only in this case the symmetry breaking $[\mathrm{U}(2) \otimes \mathrm{U}(2)]/\,\mathrm{U}(1) \to \mathrm{U}(2)/\,\mathrm{U}(1)$ is realized for $g = 0$.

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The results we have obtained above in the mean-field approximation may change when we take into account fluctuations. As discussed in ref. [10], fluctuations change the phase boundaries that are now generic curves which meet tangentially at the multicritical point. It is also possible that some of the transitions become of first order. This should happen in the case of interest of QCD. If g is the anomalous breaking of the axial U(1) symmetry, we know that for $g = 0$ the transition is of first order. Moreover, $g = 0$ should be the multicritical point, even in the presence of fluctuations, since such a point has a larger symmetry group. Since a first-order transition is generally robust under perturbations, the transition should maintain its first-order nature also for sufficiently small values of $|g|$, therefore the transition lines ending at the multicritical point are expected to be of first order sufficiently close to it. Possible phase diagrams are sketched in figure 2.

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